ANALYSIS III

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Method of characteristics

$$\begin{cases} a(x,y,u)\cdot u_x + b(x,y,u)\,u_y = c(x,y,u) \\ u(x_0(s), y_0(s)) = u_0(s) \end{cases}$$

Initial curve

$$\Gamma(s) = \begin{pmatrix} x(0,s) \\ y(0,s) \\ \hat{u}(0,s) \end{pmatrix}$$

Parameterized

$\hat{u}(t,s) := u(x(t,s), y(t,s))$

ODE, chara. equations:

$$\begin{cases} x_t(0,s) = a(x,y,u) \\ y_t(0,s) = b(x,y,u) \\ \hat{u}_t(0,s) = c(x,y,u) \end{cases}$$

Init. cond., parame. init. cond:

$$\begin{cases} x(0,s) = x_0(s) \\ y(0,s) = y_0(s) \\ \hat{u}(0,s) = u_0(s) \end{cases}$$

Transversality condition

replace x, y in ODE by init.

$$J = \begin{bmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{bmatrix} \quad \text{with } \det(J) \neq 0$$

$\Leftrightarrow$ Problem has unique solution in neighbourhood of initial curve

$\Leftrightarrow$ Points of $\Gamma(s)$ are noncharac.

$J = 0 \Rightarrow \begin{cases} \text{no} \\ \infty \end{cases}$ solutions

2nd order PDE

CLASSIFICATION: $\delta(L) = b^2 - ac$

$$L[u] = \underbrace{a u_{xx} + 2b u_{xy} + c u_{yy}}_{=\text{ High order terms}} + \underbrace{d u_x + e u_y + f u}_{\text{low order terms}} = g$$

$L_0[u]$: principal part

$$\delta(L)(x_0,y_0) = b^2(x_0,y_0) - a(x_0,y_0)\,c(x_0,y_0)$$

Derivative as LOCAL PROPERTY $\Rightarrow$ GLOBAL

☑ constant coeff

$\Rightarrow$ possible for u on $D_1 \cup D_2$

on D_1: hyperbolic

D_2: elliptic

Elliptic <0 poisson	$\nabla^2 f = 0$	BC	$\bar{D}$	smooth
parabolic 0 heat	$\partial_t f = \alpha \nabla^2 f$	BC IC	$\mathring{D}$	smooth
Hyperbolic >0 wave	$\partial_t f = c^2 \nabla^2 f$	BC IC	$\mathring{D}$	may be discontinuous

singularities travel only along the characteristics

Conservation Law

$u(x, y)$: $x \in \mathbb{R}$ position, $y \in (0, \infty)$ time

$$\begin{cases} u_y + c(u)\, u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

$c(u)$: speed

PREREQUISITES

☐ $c, u_0 \in C^1(\mathbb{R})$

☐ $c \circ u_0$ is bounded with bounded derivative

BLOW-UP: $\forall s \in \mathbb{R}$

☐ $c(u_0(s))_s \geq 0$

$\Rightarrow y_c = \infty$

CRITICAL TIME

$$\Rightarrow y_c = \inf_{c(u_0(s))_s < 0} \left\{ -\frac{1}{c'(u_0(s))\, u_0'(s)} \right\}, \quad s \in \mathbb{R}$$

$$= -\left(\inf \frac{d}{dx}\big(f'(u(x,0))\big)\right)^{-1}$$

when $c(u_0(s))_s \geq 0$, characteristics don't cross e^x

$\Rightarrow$ smooth sol. for all pos. times.

UNIQUENESS

☐ in $[0, y_c)$

$$\boxed{u(x,y) = u_0(x - c(u(x,y))\,y)}$$

u solves the implicit equation

CLASSICAL Sol

- u satisfies the PDE

$\Rightarrow$ u also satisfies the integral formulation

WEAK SOLUTION

- for discontinuity

$\rightarrow$ combi of classical sol on D_i

- Integral formulation

$$\int_a^b u(x,y_2)\,dx - \int_a^b u(x,y_1)\,dx = -\int_{y_1}^{y_2} \big[f(u(b,y)) - f(u(a,y))\big]\,dy$$

DISCON. "SHOCKS"

$\Leftrightarrow$ boundaries of $D := \bigcup_{i=1}^{n} D_i$

$\Leftrightarrow$ RH condition ☑

$$\delta'(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-}$$

The smooth curve which introduces the discontinuity

Trick: Draw out init cond, need not to consider intervals with slope $= 0$

EXPANSION

- characteristics don't cross

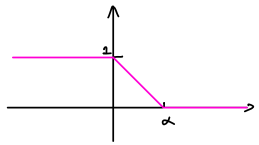
COMPRESSION

- chara. cross each other

INITIAL COND

☐ decreasing tendency

$\Rightarrow \exists$ shock



ENTROPY = "AMOUT OF INFO"

$\rightarrow$ Expectation:

Loss of information in the shock

$$\boxed{\begin{matrix} c(u^+) < \gamma' < c(u^-) \\ f'(u^+) < \gamma' < f'(u^-) \end{matrix}}$$

☑ charac. entering shock

☒ charac. coming out of a shock

☑ $\nexists$ shock $\Rightarrow$ ☑ Entropy cond

WELL-POSED

☐ Existence of a sol

☐ Uniqueness of a sol

☐ Stability ☑ small change in equation or in side conditions $\Rightarrow$ ☑ small change in sol

ILL-POSED

☐ Any well-posed cond doesn't hold

$\Rightarrow$ The problem is ill-posed

$$dA = r\, d\theta\, dr$$

COOR-TRANSFORMATION

Change of coordinates (c.o.c)

☐ A transformation

$(x,y) \mapsto (\xi, \eta) = (\xi(x,y), \eta(x,y))$

☐ Near a point (x_0, y_0)

$$\det \begin{pmatrix} \partial_x \xi & \partial_y \xi \\ \partial_x \eta & \partial_y \eta \end{pmatrix}\Bigg|_{(x_0,y_0)} \neq 0$$

$\Rightarrow$ It is a (c.o.c) near point (x_0, y_0)

Canonical form

☐ Any 2nd order PDE

☐ $u(x,y) \mapsto w(\xi, \eta)$ (c.o.c) $= w(\xi(x,y), \eta(x,y))$

$$\Rightarrow \underbrace{w_{\xi\eta}}_{w_{\bar{\xi}\bar{y}}} + \tilde{d} w_\xi + \tilde{e} w_\eta + \tilde{f} w = \tilde{g} \quad \text{hy.}$$

$$\underbrace{w_{\xi\xi}}_{w_{\bar{x}\bar{x}}} + \tilde{d} w_\xi + \tilde{e} w_\eta + \tilde{f} w = \tilde{g} \quad \text{pa.}$$

$$\underbrace{w_{\xi\xi} + w_{\eta\eta}}_{w_{\bar{x}\bar{x}} + w_{\bar{y}\bar{y}}} + \tilde{d} w_\xi + \tilde{e} w_\eta + \tilde{f} w = \tilde{g} \quad \text{elli.}$$

2D Polar

- $\Phi(r,\varphi) = (x,y) = (r\cos\varphi, r\sin\varphi)$, $r \in \mathbb{R}_{\geq 0}$, $\varphi \in (-\pi, \pi]$

- $J_\Phi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix}$

$\rightarrow \operatorname{Det}(J) = r$

1D WAVE EQUATION

☑ condition on u
☑ condition on one of the 1st derivative of u

Cauchy problem

wave speed

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,+\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

GENERAL SOL

$$\boxed{u(x,t) = \underbrace{f(x+ct)}_{\text{Backwards traveling wave}} + \underbrace{g(x-ct)}_{\text{forwards traveling wave}}}$$

D'Alembert

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + $$

NON HOMO $\left\{ \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau \right.$

NON-HOMO = $\iint_{\Delta(x_0,t_0)} F(x,t)\,dx\,dt$

ODD INITIAL DATA

☑ Extend domain of x to $\mathbb{R}$

$\Rightarrow$ ☑ D'Alembert applicable

$x^2 \to x|x| \quad x^4 \to x^3|x|$

CASE DISTINCTION:

- $x > t$: $x+t \geq 0$, $x-t \geq 0$
- $-t < x < t$: $x+t \geq 0$, $x-t < 0$
- $x \leq t$: $x+t < 0$, $x-t \leq 0$

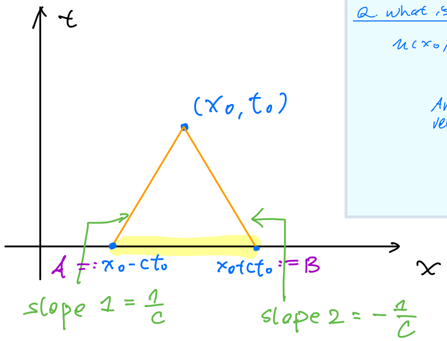
DOMAIN OF DEPENDENCE at (x_0,t_0)

$\underbrace{[x_0-ct_0,\ x_0+ct_0]}_{\text{Domain of dep.}} \subset \underbrace{\Delta_{(x_0,t_0)}}_{\text{charac. triangle}}$

Lemma

☑ Init. cond smooth on $[x_0-ct_0, x_0+ct_0]$

$\Rightarrow$ ☑ u is smooth in the chara. triangle $\Delta_{(x_0,t_0)}$



Q. what is $u(x_0,t_0)$?

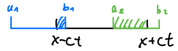
$u(x_0,t_0)$ = Average of f at A and B
+
Average velocity: Average of g from A to B

$\times$ t_0

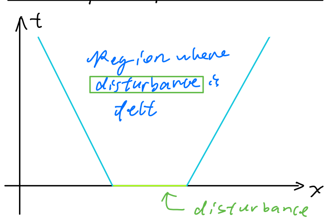
REGION OF INFLUENCE

- a point set
- Geg. fixed interval $I := [a,b]$
  Ges. Data influenced by init. data on I

$$[x_0-ct_0,\ x_0+ct_0] \cap [a,b] \neq \emptyset$$
$$\begin{cases} x - ct \leq b \\ x + ct \geq a \end{cases} \wedge$$



Finite speed of a disturbance



Typ: Find singularities

$x+ct = \text{const}$ $\begin{cases} x+ct = a \\ x+ct = b \end{cases}$

$x-ct = \text{const}$ $\begin{cases} x-ct = a \\ x-ct = b \end{cases}$

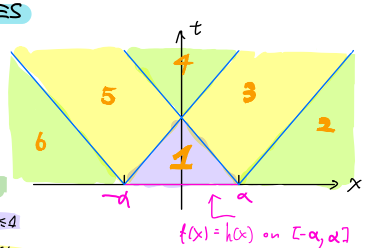
Typ: Parity of sol

☐ Non-homo – even/odd
☐ $u(x,0)$ – even/odd
$\Rightarrow$ $u(x,y)$ – even/odd

CHARACTERISTIC LINES

$$f(x) = \begin{cases} h(x), & x \in [-\alpha,\alpha] \\ 0, & \text{else} \end{cases}$$

$g(x) = 0$



2, 4, 6	$u = 0$	$\lvert x-t\rvert > 1 \wedge \lvert x+t\rvert > 1$
1	$u = h(x)$	$\lvert x-t\rvert \leq 1 \wedge \lvert x+t\rvert \leq 1$
3, 5	$u = \frac{h(x\pm t)}{2}$	$\lvert x\pm t\rvert \leq 1 \wedge \lvert x\mp t\rvert > 1$

5: $u = \frac{h(x+t)}{2}$ $\quad |x+t| \leq 1 \wedge |x-t| > 1$

3: $u = \frac{h(x-t)}{2}$ $\quad |x-t| \leq 1 \wedge |x+t| > 1$

M.O.C for wave

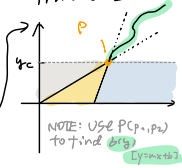
① Find ODE from characteristics & solve

② Obtain expression of $\tilde{u}(0,s)$ with case distinctions based on s

③ Invert the expressions
From: $x(t,s) = s + \alpha$
To: $s = x - \alpha$

④ change the case entries to
$x \leq \cdots$
$\cdots < x < \cdots$
$x \geq \cdots$

From this draw



ENERGY METHOD

$$\boxed{E(t) = \int_0^L \left(w_t(t,x)\right)^2 + c^2\left(w_x(t,x)\right)^2 dx}$$

$$\frac{d}{dt}E(t) = \int_0^L (2w_t w_{tt} + 2c^2 w_x w_{xt})\,dx$$

$$= 2\int_0^L (w_t w_{tt} - c^2 w_{xx} w_t)\,dx + \underbrace{\left[2c^2 w_x w_t\right]_0^L}_{=0}$$

$w_t(w_{tt} - c^2 w_{xx}) = 0$

By init. cond from transformed problem

①: Define $w := u_1 - u_2$, rewrite the prob.

②: Derive the energy func

③: $E(t)$ must be const
$\Rightarrow$ w_x, w_t thus w const (*)

④: $w(x,t)$ must be zero since its initial value is zero and (*)

⑤: $w := u_1 - u_2$ leads to uniqueness

Uniqueness – Wave Equation

☐ Initial value problem for wave equation

Non-homo

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & 0 < x < \ell \\ u(x,0) = f(x), & 0 \leq x \leq \ell \\ u_t(x,0) = g(x), & 0 \leq x \leq \ell \\ u(0,t) = u(\ell,t) = 0, & t > 0 \end{cases}$$

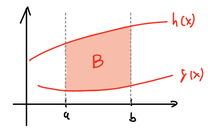
$\exists$ solution $u(x,t)$ (Ene. method)

$\Rightarrow$ u is unique solution

NORMAL BEREICH

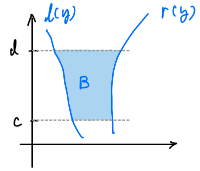
TYP I: $B = \{(x,y) : a \leq x \leq b \wedge g(x) \leq y \leq h(x)\}$

$$\boxed{\iint_B f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy\right) dx}$$



TYP II: $B = \{(x,y) : c \leq y \leq d \wedge l(y) \leq x \leq r(y)\}$

$$\iint_B f(x,y)\,dx\,dy = \int_c^d \left(\int_{l(y)}^{r(y)} f(x,y)\,dy\right) dx$$



Orthonormality

☐ $n, m \in \mathbb{N} \cup \{0\}$

1) $\displaystyle\int_{-L}^{L} \cos\left(\frac{n\pi}{L}t\right)\cdot\cos\left(\frac{m\pi}{L}t\right)dt = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases}$

2) $\displaystyle\int_{-L}^{L} \sin\left(\frac{n\pi}{L}t\right)\cdot\sin\left(\frac{m\pi}{L}t\right)dt = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases}$

3) $\displaystyle\int_{-L}^{L} \cos\left(\frac{n\pi}{L}t\right)\cdot\sin\left(\frac{m\pi}{L}t\right)dt = 0 \quad \forall n,m$

☐ f is $2L$-periodic

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}t\right) + b_n\sin\left(\frac{n\pi}{L}t\right)\right)$$

Coefficients

Complex

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}t}$$

FOURIER SERIES

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)dt, \quad m \geq 0$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{m\pi}{L}t\right)dt, \quad m \geq 0$$

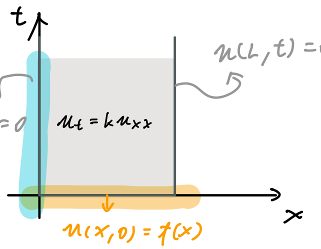
$$c_m = \frac{1}{2L}\int_{-L}^{L} f(t)\,e^{-i\frac{m\pi}{L}t}\,dt, \quad m \in \mathbb{Z}$$

SEPARATION OF VARIABLES

$$u(x,t) = X(x)\,T(t)$$

$X : [0,L] \to \mathbb{R}$

$T : [0,+\infty) \to \mathbb{R}$



ANSATZ (HOMO)

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Dirichlet condition

$u_t - k u_{xx} = 0 \qquad (x,t) \in (0,L) \times (0,+\infty)$

General Solution:

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{C_n \cdot \sin(\sqrt{\lambda_n}\, x)}_{X_n(x)} \cdot e^{-k\lambda_n t}$$

$\lambda = \left(\frac{n\pi}{L}\right)^2$

Von Neumann condition

$u_{tt} - c^2 u_{xx} = 0 \quad , \quad (x,t) \in [0,L] \times (0,+\infty)$

$$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \underbrace{\cos\left(\frac{n\pi}{L}x\right)}_{X_n(x)} \left[A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)\right]$$

ANSATZ (INHOMO)

$$v(x,t) = \sum_{n=0}^{+\infty} T_n(t)\, X_n(x)$$

WANTED EXPR

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{L}x\right) \quad \text{D.B.C}$$

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t) \cos\left(\frac{n\pi}{L}x\right) \quad \text{V.N.B.C}$$

⓪. Check if B.C are homo, if not

$v(x,t) := u(x,t) - \underbrace{w(x,t)}_{\text{what satisfies B.C}}$

① Plug inhomo Ansatz into the equation

② Obtain case distinction for X_n terms, $n \in \mathbb{N}$ → shall finally be one of wanted expr.

③ Case dis. by RHS of additional conditions eg. $n = \alpha, \beta$

→ eg. case $n = \alpha$
$$\begin{cases} T_\alpha''(t) & \text{[From equation]} \\ T_\alpha(J) = [\text{From } u(x,J) = \dots] & \Rightarrow T_\alpha(t) = \dots \\ T_\alpha'(k) = [\text{From } u_t(x,k) = \dots] \end{cases}$$

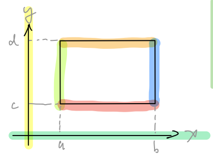
when $n = 0$
☑ for const terms

case $n \neq \alpha, \beta$

case $n = \beta$

④. Obtain $u = v + w$

SPLITTED LAPLACE



$$\text{ANSATZ: } \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda \in \mathbb{R}$$

⓪. ☑ Nonhomo?
⇒ Rewrite!
Let $v := u + f(x,y)$

$$\begin{cases} \Delta u = f(x,y) & ,(x,y) \in (a,b) \times (c,d) \\ u(a,y) = \dots & , y \in [c,d] \\ u(b,y) = \dots & , y \in [c,d] \\ u(x,c) = \dots & , x \in [a,b] \\ u(x,d) = \dots & , x \in [a,b] \end{cases}$$

u_1 – y changes

u_2 – x changes

Neumann Problem

$$\begin{cases} \Delta u(x,y) = \rho(x,y) & ,(x,y) \in D \\ \partial_\nu u(x,y) = g(x,y) & ,(x,y) \in \partial D \end{cases}$$

↳ Normal vector

Equilibrium cond

$$\int_{\partial D} \underbrace{g(x(s), y(s))}_{\text{Heat flux}}\, ds = \int_D \underbrace{\rho(x,y)}_{\text{Temperature production}}\, dx\, dy$$

⇒ ☑ Von Neumann prob has a solution

$$u(x,y) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n}(y-c)) \left[A_n \sinh(\sqrt{\lambda_n}(x-a)) + B_n \sinh(\sqrt{\lambda_n}(x-b))\right]$$

$$X_n(x) = \alpha_n \sinh(\sqrt{\lambda_n}(x-a)) + \beta_n \sinh(\sqrt{\lambda_n}(x-b))$$

$$u_1(x,y) = \sum_{n=1}^{\infty} \left[A_n \sinh\left(\frac{n\pi}{L}(x-a)\right) + B_n \sinh\left(\frac{n\pi}{L}(x-b)\right)\right] \cdot \sin\left(\frac{n\pi}{L}(y-c)\right)$$

$L = d - c$

$$u_2(x,y) = \sum_{n=1}^{\infty} \left[C_n \sinh\left(\frac{n\pi}{L}(y-a)\right) + D_n \sinh\left(\frac{n\pi}{L}(y-b)\right)\right] \cdot \sin\left(\frac{n\pi}{L}(x-c)\right)$$

$L = b - a$

Gradient of $u(x,y,z)$

$$\nabla u := (u_x, u_y, u_z)$$

Laplacian of u

$$\Delta u := u_{xx} + u_{yy} + u_{zz}$$

Laplacian (Polar)

$$\Delta u = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}$$

$u(x(r,\theta), y(r,\theta))$
$= u(r\cos\theta, r\sin\theta)$
$= w(r,\theta)$

POLAR

$\Delta u = 0 \iff w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0$

$$w(r,\theta) = R(r)\,\Theta(\theta) \quad \text{Ansatz}$$

$$(\star) \quad -\frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{r^2 R''(r) + r R'(r)}{R(r)} \overset{(+)}{=} \lambda$$

Periodicity
☑ Θ and Θ' are periodic in $\theta \in [0, 2\pi]$
☑ $\Theta'' = -\lambda\Theta$ $(\star)$
⇒ Θ'' is periodic

Did it? → ☐ Expand to ODE ⊕ → ☐ EXTRA COND.

$$w(r,\theta) = \sum_{n=1}^{\infty} R_n(r)\, \Theta_n(\theta)$$

Radius

$$R_n(r) = \begin{cases} C_0 + D_0 \log r, & n = 0 \\ C_n r^n + D_n r^{-n}, & n \neq 0 \end{cases}$$

If $(0,0)$ is contained
⇒ $D_0 = D_n = 0$ [singularity]

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

CHECK!

EXTRA COND.
☑ $\Theta(0) = \Theta(2\pi)$
☑ $\Theta'(0) = \Theta'(2\pi)$

⇒ u is classical sol inside D
u is C^2

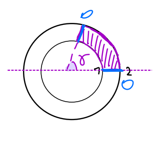
SIMPLIFIED – ☑ $(0,0) \in D$

$$w(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n \underbrace{\left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]}_{\Theta_n(\theta)}$$

GENERAL – ☑ $(0,0) \notin D$

$$w(r,\theta) = E + F \log r + \sum_{n=1}^{\infty} \left[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) + C_n r^{-n} \cos(n\theta) + D_n r^{-n} \sin(n\theta)\right]$$

ANNULAR SECTOR



$$\Theta_n(\theta) = A_n \sin\left(\frac{n\pi}{\gamma}\theta\right)$$

$$R_n(r) = C_n r^{\sqrt{\lambda}} + D_n r^{-\sqrt{\lambda}} = C_n r^{\frac{n\pi}{\gamma}} + D_n r^{-\frac{n\pi}{\gamma}}$$

$$\lambda_n = \left(\frac{n\pi}{\gamma}\right)^2$$

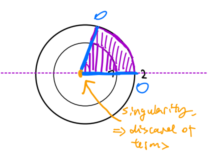
Goal: $R_n(r)$

STEP 0: Derive Ansatz in $(\star)$

Ansatz: $R(r) = c \cdot r^\alpha$

STEP 1: solve for α ⇒ $R_n(r)$ as sum of two cases of α

CIRCULAR SECTOR



$$w(r,\theta) = \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}} + B_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{-\frac{n\pi}{\gamma}}\right]$$

SAME FORMULAR WITHOUT THIS FOR CIRCULAR SECTOR

MAXIMUM PRINCIPLE

Hyperbolic > 0 — wave ☒

may be discontinuous

NOTE

- ☐ $u \in C^2$ on D open
- ☐ $\max u = u(x_0)$, $x_0 \in D$

$\Rightarrow \begin{cases} Du(x_0) = 0 \\ D^2 u(x_0) \leq 0 \end{cases}$?

P: Show contra.

STEP 1: Show $\boxed{\max_{\bar D} u_\varepsilon = \max_{\partial D} u_\varepsilon}$

$\Delta u_\varepsilon(\bar x, \bar y) \leq 0 \to$ local max

$\Delta u_\varepsilon(\bar x, \bar y) > 0 \to$ harmonic $\left.\begin{matrix}\Delta u \\ + \\ \varepsilon > 0\end{matrix}\right\} \Rightarrow > 0$

STEP 2: Obtain $\max_{\partial D} u + \varepsilon C$

Ansatz: $u \leq u_\varepsilon \,\Big|\, \max_{\bar D}$

STEP 3: Let $\varepsilon \to 0$

WEAK MAX. PRINCIPLE

- ☐ $D \subseteq \mathbb{R}^n$ bounded
- ☐ $u(x,y) \in C^2(D) \cap C(\bar D)$
- ☐ u is harmonic on D

$\Rightarrow \boxed{\max_{\bar D} u = \max_{\partial D} u}$

P. ② Use

② use in the middle

(COR). $\Rightarrow \boxed{\min_{\bar D} u = \min_{\partial D} u}$

• Typ Find min. value $\min_{\partial D} [u(x,y)]$

① Parameterize
- Replace (x,y) using variable z
- change ∂D to expression of $z \in [\dots]$

② Reduce expression and transform back to expr. using only x or y

③ Find max/min normally

Typ: General Uniqueness

1. Let $v := u_1 - u_2$, list out new prob
2. Take point (x_0, y_0)

Goal: Show $\max_{\bar D} v = \min_{\bar D} v = 0$

2.1 Assume $\max_{\bar D} v = M > 0$
$\Rightarrow v(x_0, y_0) = M > 0$
$\to 0 = \Delta v(x_0,y_0) - k v(x_0,y_0) \leq -kM > 0$ ↯

2.2. Assume $\min_{\bar D} v = M < 0$
$\Rightarrow v(x_0, y_0) = M < 0$
$\to 0 = \Delta v(x_0,y_0) - k v(x_0,y_0) \geq -kM > 0$ ↯

MEAN VALUE PRINCIPLE

- ☐ u is harmonic on D
- ☐ $B_R(x_0, y_0) \subset D$ ↳ Ball of radius R

$\Longleftrightarrow$

• Typ Eval of single point in D

$$\boxed{\begin{aligned} u(x_0, y_0) &= \frac{1}{2\pi R} \int_{\partial B_R(x_0,y_0)} u(x(s), y(s))\, ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\, d\theta \end{aligned}}$$

STRONG MAX. PRINCIPLE

- ☐ u harmonic on D
- ☐ $D \subseteq \mathbb{R}^2$ is an open, connected set
- ☐ u attains its max./min. at $x_0 \in \mathring{D}$

$\Rightarrow u$ is const on D

POISSON

Elliptic < 0 — Poisson ☑

P. Both $u^1 - u^2$ solve the problem $u^2 - u^1$

• Uniqueness

- ☐ $D \subset \mathbb{R}^2$ bounded
- ☐ Dirichlet problem
$\begin{cases} \Delta u = f, & \text{in } D \\ u = g', & \text{in } \partial D \end{cases}$

$\Rightarrow$ There is at most one solution $u \in C^2(D) \cap C(\bar D)$

• Greatest Difference

- ☐ $D \subseteq \mathbb{R}^2$ bounded
- ☐ Problem 1 $\begin{cases} \Delta u_1 = 0, & \text{on } D \\ u_1 = g_1, & \text{on } \partial D \end{cases}$ Problem 2 $\begin{cases} \Delta u_2 = 0, & \text{on } D \\ u_2 = g_2, & \text{on } \partial D \end{cases}$

with solution u_1, u_2

- ☐ $u_1, u_2 \in C^2(D) \cap C(\bar D)$

$\Rightarrow \boxed{\max_{\bar D} |u_1 - u_2| = \max_{\partial D} |g_1 - g_2|}$

HEAT

Parabolic $= 0$ — heat ☑ MAX. PRINCIPLE

- ☐ Problem
$u_t = k\Delta u$, $(t,x) \in \underbrace{[0,T]}_{\text{TIME}} \times \underbrace{D}_{\text{spatial domain}}$, $k > 0$, $Q_T = [0,T] \times D$
with solution u
- ☐ $D \subseteq \mathbb{R}^2$ is bounded

$\Rightarrow u$ achieves its max. and min on $\underbrace{\{\{0\} \times D\} \cup \{[0,T] \times \partial D\}}_{\partial_p Q_T \text{ "parabolic boundary"}}$

• Uniqueness
- ☐ Dirichlet problem
$\begin{cases} u_t - k\Delta u = f, & \text{in } Q_T \\ u(0, \underline{x}) = g(\underline{x}), & \text{in } D \\ u(t, \underline{x}) = h(\underline{x}), & \text{in } [0,T] \times \partial D \end{cases}$

(*) $\Rightarrow$ has unique solution

(*) By max. principle

TRIGONOMETRY

$\cosh^2(x) - \sinh^2(x) = 1$

$\sin(z) = \frac{1}{2i}[e^{iz} - e^{-iz}]$ $\quad \cos(z) = \frac{1}{2}[e^{iz} + e^{-iz}]$

$\sinh(x) = \frac{1}{2}[e^x - e^{-x}]$ $\quad \cosh(x) = \frac{1}{2}[e^x + e^{-x}]$

$\sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$ $\quad \sin^3(x) = \frac{1}{4}[3\sin(x) - \sin(3x)]$

$\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$ $\quad \cos^3(x) = \frac{1}{4}[3\cos(x) + \cos(3x)]$

$\sin^4(x) = \frac{1}{8}[\cos(4x) - 4\cos(2x) + 3]$

$\cos^4(x) = \frac{1}{8}[\cos(4x) + 4\cos(2x) + 3]$

$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$

$\sin(x)\cos(y) = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

$\sin(x)\cos(x) = \frac{1}{2}[\sin(2x)]$

$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

$\sin(x+y)\sin(x-y) = \cos^2(y) - \cos^2(x) = \sin^2(x) - \sin^2(y)$

$\cos(x+y)\cos(x-y) = \cos^2(y) - \sin^2(x) = \cos^2(x) - \sin^2(y)$

INTEGRAL

$f(x)$	$F(x) + C$
$\frac{1}{x}$	$\ln|x|$
α^x	$\frac{\alpha^x}{\ln|\alpha|}$
$\frac{1}{\cos^2 x}$	$\tan x$
$\frac{1}{\sin^2 x}$	$-\cot x = -\frac{1}{\tan x}$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\frac{1}{1+x^2}$	$\arctan x$
$\frac{1}{\cosh^2 x}$	$\tanh x$
$\frac{1}{\sinh^2 x}$	$-\coth x$

$$\boxed{\int_a^b f(x) G(x)\, dx = [F(x) G(x)]_a^b - \int_a^b F(x) g(x)\, dx}$$

$\cos^2 x$	$\frac{\cos x \sin x + x}{2}$
$\sin^2 x$	$\frac{x - \cos x \sin x}{2}$
$\cos x \sin x$	$\frac{\sin^2 x}{2} = -\frac{1}{2}\cos^2 x$

$$\boxed{\begin{aligned} &\int_0^{2\pi/\omega} \cos(k\omega x)\sin(l\omega x)\, dx = 0 \\ &\int_0^{2\pi} \cos(x)\, dx = \int_0^{2\pi} \sin(x)\, dx = 0 \\ &\int_0^{2\pi} \cos^3(x)\, dx = \int_0^{2\pi} \sin^3(x)\, dx = 0 \\ &\int_0^{2\pi} \cos^4(x)\, dx = \int_0^{2\pi} \sin^4(x)\, dx = \frac{3\pi}{4} \end{aligned}}$$